Broad-Band Quarter-Wave Plates*

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Summary—**Analytical expressions are given for the propagation constant for the two orthogonal dominant modes in a square waveguide loaded with a centered slab of dielectric. These are combined to find the differential phase shift in a loaded section. Solutions of these equations are given for polystyrene which enables one to design broad-band quarter wave plates. The problem of transforming from unloaded to loaded guide is discussed and two solutions are given and another suggested. Experimental results are given and it is found that the work in square guide carries over to circular guide almost intact. It is shown that quarter-wave plates may easily be designed to cover a normal waveguide bandwidth.**

Introduction

CIRCULARLY-polarized waves are being used to a great extent to exploit the possibilities of microwave techniques. To convert the microwave energy from linear to circular polarization or vice versa, some form of quarter-wave plate is necessary. A quarter-wave plate creates a relative phase delay, or differential phase shift, of 90° between two spatially orthogonal modes. A detailed description of the operation and the uses of quarter-wave plates as well as some of the early work is described elsewhere.[1,2] Among the various forms there is good reason to believe that the dielectric slab in a waveguide is one of the more suitable constructions for broadbanding. The reasoning behind this belief comes from the nature of the dispersiveness of a waveguide loaded as in Fig. 1 with a slab of dielectric. It will be seen later that the differential phase shift which is the difference in dispersiveness for the two orthogonal modes is not a monotonically increasing function of frequency as one might expect, but is actually a second or third-order curve in the frequency range of interest.

Although a quarter-wave plate can always be constructed by trial and error, this procedure gives a bandwidth which cannot be predicted in advance or be systematically improved. Thus, a method of calculating the differential phase shift for two orthogonal linearly-polarized waves in a partially dielectric-loaded square waveguide is presented. The results for square waveguide should carry over to circular waveguide quite well and this is verified experimentally.

The normal modes of a rectangular waveguide, with a longitudinal slab of dielectric in it, are longitudinal section electric and magnetic modes, or for brevity LSE and LSM modes, respectively. A detailed mathematical treatment of these modes, starting with Maxwell's equations and the boundary conditions, was carried out

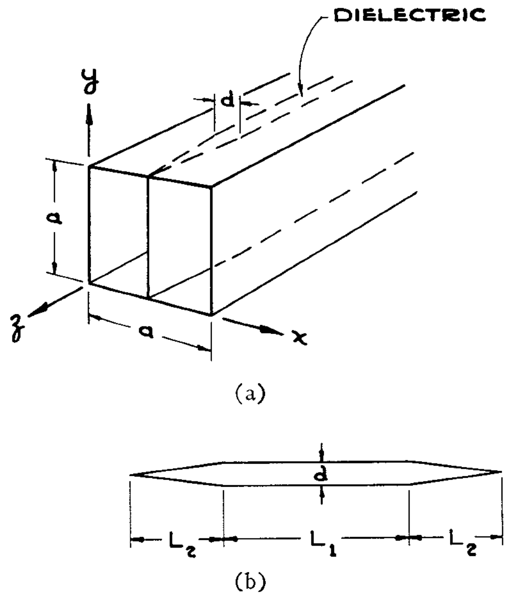

Fig. 1—(a) Dielectric loaded square waveguide; (b) top view of tapered quarter-wave plate.

and follows the method of Pincherle.[3] The results of the mathematics for the two modes of interest in quarter-wave plate design are given. These are the modes that become TE_{10} and TE_{01} modes when the dielectric slab is reduced to zero thickness. Method of solution of the characteristic equations is graphical. Using these graphical solutions, design of quarter-wave plates and experimental results for one of the designs are discussed.

Theory

For the purpose of constructing a quarter-wave plate, one is interested in the differential phase shift in wavelengths,

$$\Phi = L\left(\frac{1}{\lambda_{g1}} - \frac{1}{\lambda_{g2}}\right) \tag{1}$$

where L is the length of the plate and λ_{g1}, λ_{g2} are the guide wavelengths for two orthogonal linear polarizations. This differential phase shift must be maintained at a constant value, 0.250, over a given frequency range. For this purpose, a dielectric loaded square waveguide, as shown in Fig. 1(a), propagating the two lowest modes is now considered. For each mode, one must solve an equation which is due to the boundary conditions at the dielectric-air interfaces to obtain a quantity which is called the transverse eigenvalue. This quantity can then be substituted into an equation relating the propagation constants, or, as written here, relating the various wavelengths. For the LSE mode corresponding to the empty

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[1] A. G. Fox, "An adjustable waveguide phase changer," Proc. IRE, vol. 35, pp. 1489–1498; December, 1947.

[2] G. C. Southworth, "Principles and Applications of Waveguide Transmission," D. Van Nostrand Co., Inc., New York, N. Y., p. 325; 1950.

[3] L. Pincherle, "Electromagnetic waves in metal tubes filled longitudinally with two dielectrics," *Phys. Rev.*, vol. 66, pp. 118–130; September, 1944.

waveguide TE_{10} mode ($E_x=0$), the boundary condition relation to be solved is[4]

$$\frac{\tan x_1}{x_1} = K \frac{\operatorname{ctn} \sqrt{K^2 x_1^2 + C^2}}{\sqrt{K^2 x_1^2 + C^2}} \tag{2}$$

where

$K = d/(a-d)$,
$C = \pi d(\epsilon - 1)^{1/2}/\lambda$,
d = thickness of the dielectric,
ϵ = relative dielectric constant of the slab,
λ = free space wavelength, and
x_1 = variable related to the transverse eigenvalue.

The lengths have been normalized to the guide width a to make the solutions easily adaptable to any waveguide size. The method of solution of (2) is graphical and, as one can see, there is an infinity of solutions due to the periodic functions involved. The solution for the desired mode is that for which $|x_1|$ is smallest. This result is then substituted into the propagation constant equation

$$\nu_1^2 = \nu_0^2 - \eta_1^2 \tag{3}$$

where

$\eta_1 = x_1 a/\pi(a-d)$
$\nu_0 = a/\lambda$
$\nu_1 = a/\lambda_g$ = phase shift per unit length in units of wavelengths/guide widths.

For the LSM mode corresponding to the empty waveguide $TE_{01}(E_y=0)$ mode, the boundary condition relation to be solved is

$$x_2 \tan x_2 = -\frac{1}{K\epsilon} \sqrt{K^2 x_2^2 + C^2} \tan \sqrt{K^2 x_2^2 + C^2}. \tag{4}$$

The method of solution is again graphical and, as before, the desired solution is that for which $|x_2|$ is smallest. In solving (2) and (4) it must be kept in mind that x_1 and x_2 may be either real or imaginary. The solution for x_2 is then substituted into the propagation constant

$$\nu_2^2 = \nu_0^2 - \eta_2^2 - \tfrac{1}{4} \tag{5}$$

in order to find ν_2. All quantities here are defined the same as for (2) and (3) except the subscripts are changed to 2.

The relations given above are exact, no approximations have been made. All of the numerical results to be described are for polystyrene, $\epsilon = 2.55$, and were obtained by graphical means. The propagation constants were found to three or four significant figures to obtain their difference to three significant figures. The solutions of (3) and (5) are used to find the differential phase shift per unit length a

$$\phi = \nu_1 - \nu_2. \tag{6}$$

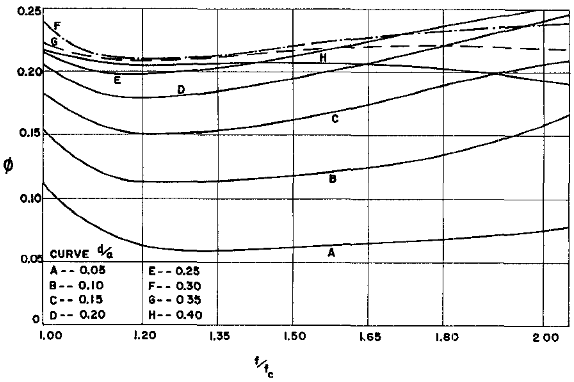

Fig. 2—Differential phase shift as a function of frequency for polystyrene plates, $\epsilon = 2.55$, of various thicknesses, d.

The differential phase shift is found for a number of frequencies and plotted in Fig. 2 as a function of $f/f_c = 2\nu_0$. Here f is the frequency parameter and f_c is the cutoff frequency of the empty waveguide. From this information, quarter-wave plates may be designed.

In actual practice one must match the ends of the plate to the empty waveguide. This is most easily done in a predictable manner by tapering the thickness d at both ends as shown in Fig. 1(b). In order to predict the effect of these tapers, many curves are needed for different plate thicknesses. A family of these are shown in Fig. 2 with d as a parameter. These are cross-plotted in Fig. 3 with d as the variable and f/f_c as the parameter for the family. From Fig. 3 the effect of tapers can be estimated. These curves are integrated and the results plotted in Fig. 4 so that the effect of linear tapers up to any thickness d can be found immediately. In using these curves, the normalization of all quantities must be remembered and, in particular, it must be noticed that the differential phase shift is in units of 2π radians per unit width a of the waveguide.

Design of Quarter-Wave Plates

To design a complete plate, the sum of the differential phase shifts due to the two tapers and the flat midsection must add up to 0.250 over frequency band of interest. This is done by using Figs. 3 and 4 to pick a value of d that gives a reasonably flat differential phase shift for a combination of the tapers and flat midsection. Then the sum of the differential phase shifts are multiplied by an appropriate constant to make the total shift

$$\Phi = 2L_{\text{taper}}\,\theta + L_{\text{midsection}}\text{-}\phi = 0.250 + \Delta\Phi$$

where $\Delta\Phi$ is the error in wavelengths. The error $\Delta\Phi$ in differential phase shift gives rise to elliptical polarization instead of circular when the plate is used. Axial voltage ratio defined as the ratio of the electric field along the major axis of the ellipse to that along the minor axis is

$$A = \tan\left(|\Delta\Phi| + \tfrac{1}{4}\right)\pi \tag{7}$$

[4] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., vol. 8, pp. 386–387; 1948. Eq. (49), p. 387, appears to be a typographical error. It should have *-ctn* replaced by *tan*. As (49) is written together with (50) it would give the propagation constant for the next higher order LSE mode which is the one that evolves from the empty waveguide TE_{20} mode as dielecrtic is introduced into the waveguide. The curves on p. 386 are essentially correct except for small irregularities which are not shown. These irregularities are not important for most work. However, they are extremely important for quarter-wave plate work where the small differences between these curves and another similar set are the differential phase shifts.

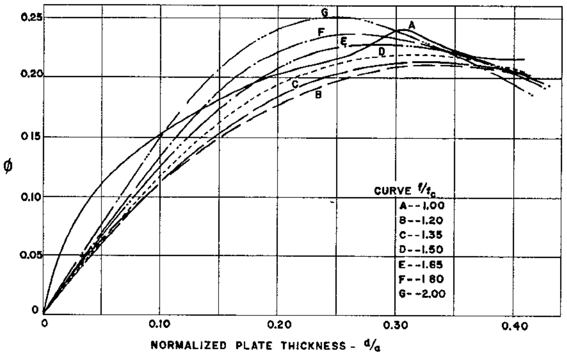

Fig. 3—Differential phase shift as a function of plate thickness for a polystyrene dielectric, $\epsilon=2.55$, for various frequencies.

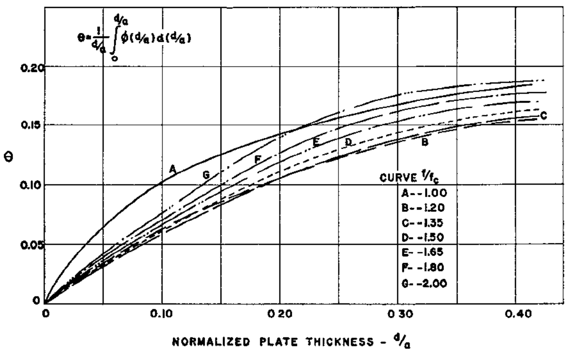

Fig. 4—Differential phase shift for a tapered plate of polystyrene as a function of the final thickness d of the taper for various frequencies.

and is plotted in Fig. 5. In practice the power going into the undesired circular polarization manifests itself as a transmission loss given by

$$P_{\text{loss}}/P_0 = \sin^2 \pi\Delta\Phi. \tag{8}$$

This loss is expressed as an insertion loss in decibels on the right-hand ordinate scale of Fig. 5.

To illustrate the design procedure, an example will be carried out. Fig. 2 indicates that $d/a=0.40$ (curve H) gives a frequency response for a flat slab of dielectric which is nearly constant with the differential phase shift dropping slightly at higher frequencies. This is a good choice since Fig. 4 shows that the differential phase shift for tapered slabs of dielectric increases with increasing frequency for all thicknesses for f/f_c between 1.2 and 2.0. Choosing several values of L_1/L_2, the ratio of the flat length to the tapered length, and using $d/a=0.40$, the following results are obtained for the usual bandwidth $1.2<f/f_c<2.0$.

$$L_1 / L_2 = 4, \quad L_1 = 0.874a, \quad L_2 = 0.219a,$$
$$\Delta\Phi = \pm 0.003.$$

Loss due to incorrect circular polarization ≤ 0.0005 db.

$$L_1/L_2 = 0, \quad L_1 = 0, \quad L_2 = 0.731a,$$
$$\Delta\Phi = \pm 0.024.$$

Loss due to incorrect circular polarization ≤ 0.025 db.

A practical aspect of the problem now arises; namely, the reflections from the ends of the dielectric. For the

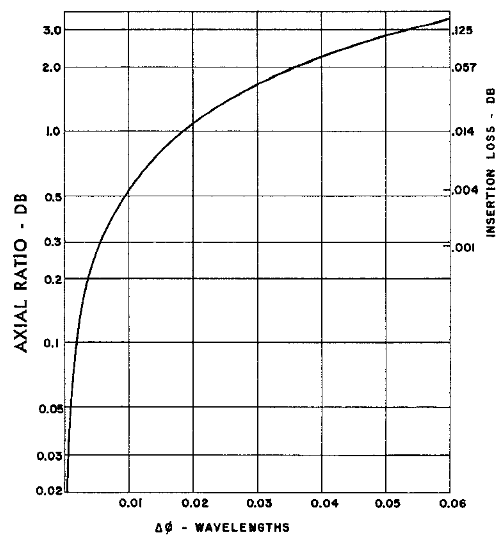

Fig. 5—Axial ratio of output from quarter-wave plate and insertion loss due to incorrect circular polarization as a function of the error in differential phase shift.

case of $L_1/L_2=4$, L_2 is so small that the tapers present an angle of almost 45° to the incident wave and will produce so much reflection that the device would be useless as a quarter-wave plate. On the other hand, the taper angle for $L_1=0$ is about 15° and the loss due to the incorrect circular polarization is still very small. Thus, from a practical standpoint, the choice $L_1=0$ would be much better than $L_1/L_2=4$ for this thickness. A quarter-wave plate was made with $L_1=0$, and the results will be discussed later.

In light of the reflection problem, perhaps it is better to choose a smaller value of d. Taking $d/a=0.2$, $L_1=0$ then $L_2=1.042a$, $\Delta\Phi=\pm 0.031$, and loss due to incorrect circular polarization ≤ 0.04 db. Taper angle for this is about $5\frac{1}{2}$° which should produce negligible reflections.

Experimental Results

A quarter-wave plate was built of polystyrene, $\epsilon=2.55$ with $d/a=0.40$ and $L_1=0$ to cover the range $1.0<f/f_c<2.0$. The theoretical prediction and the experimental results are shown in Fig. 6 for comparison. The experimental points were found by measuring the phase of the two orthogonal waves separately and subtracting them to obtain the differential phase shift. The experimental points represent a frequency range of 7.6 to 12.5 kmc. The only substantial disagreement with the theoretical curve is seen to be at the extremes of the curve. At these points the frequency is outside the recommended operating range of the waveguide components used. The departure of the remaining points from the theoretical curve may be attributed to internal reflections in the quarter-wave plate. These are not calculable by any simple analysis, but do not seem to be too bothersome for the tapers used.

In using such a plate, it must be made certain that equal incident amplitudes are obtained in each of the two orthogonal directions. First, the plate must be oriented at exactly 45° to the polarization of the incident linearly-polarized wave. Then, the reflections from one of the

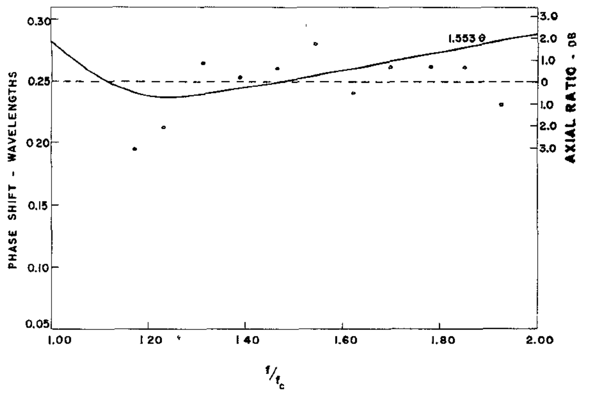

Fig. 6—Theoretical curve and experimental points for a polystyrene taper 1.553 guide widths long and 0.40 guide width thick at its thickest point.

two orthogonal directions of polarization must not be stronger than from the other. Effectively this destroys the condition of equal amplitude through the plate for the two polarizations. In measurements on circularity of the transmitted wave, this effect has been very confusing at times. In general, the only solution to this problem is to make the tapers gradual enough so that these reflections are not detrimental to the operation of the quarter-wave plate. From this standpoint, a better practical solution may well be the design given at the end of the last section for which the taper angle is only $5\frac{1}{2}°$ and the loss is less than 0.04 db. The same problems can be caused also by dielectric loss. However, polystyrene does not have enough loss to be significant in this regard. A quarter-wave plate designed for square waveguide was experimentally checked in square guide and then rechecked in a circular guide of diameter just slightly larger than the width of the square guide. Its performance in circular guide over the frequency band 8.0 to 12.5 kmc was slightly better than in square guide. This confirmed the opinion that there should be very little difference between the performance of a quarter-wave plate in square guide and in circular guide.

Another design which departs from the tapers and uses step transitions is shown in Fig. 7. The transformers on each end were arrived at by considering the match for the TE_{10} mode only since this mode is much more perturbed by the dielectric plate than the TE_{01} mode. The impedances of the steps were calculated on a power-voltage basis by considering the wave guide to be filled with some effective dielectric constant. The effective dielectric constant is found from the relation

$$\lambda_g = \lambda\left[\epsilon_{\text{eff}} - (\lambda/2a)^2\right]^{-1/2},$$

and used in the power-voltage impedance equation

$$Z_{P,v} = 2(Z_0/\sqrt{\epsilon_{\text{eff}}})b\lambda_g/a\lambda$$

which gives the impedance relation that was used,

$$Z_{P,v} = 2Z_0(\lambda_g/\lambda)^2[1 + (\lambda_g/2a)^2]^{-1/2}$$

where the guide wavelengths were found from existing curves.[4] The appropriate quarter wavelengths were also found from these curves. It is seen in Fig. 7 that this quarter-wave plate turned out quite well having an axial

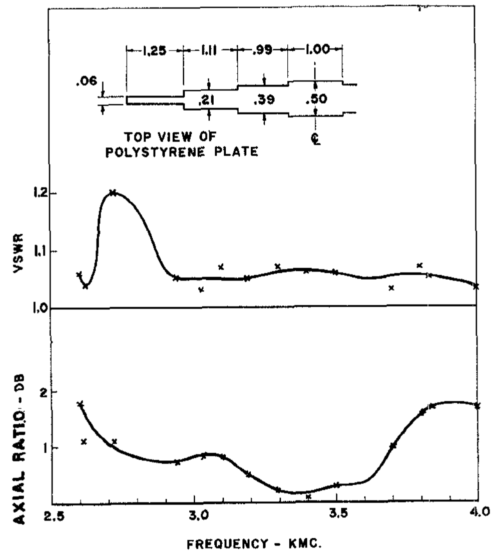

Fig. 7—Top view of step tapered polystyrene quarter-wave plate. VSWR and axial ratio of output circular polarization are given as functions of frequency. Waveguide was circular with an inside diameter of 3.125 inches.

ratio under 2 db and a vswr under 1.2 to 1 over S band. A study that will facilitate the design of step transitions is now being carried out by the author and others. A study that should be undertaken is that of the differential phase shift for the two orthogonal modes for a dielectric plate that is tapered in the opposite plane to the taper of Fig. 1(b). For the same taper length this type of taper gives much smaller reflections than the tapers described here. Unfortunately, calculations for this taper seem impractical to carry out. Furthermore, this taper will not carry high peak powers since, unlike the previous two tapers described, it has a region where a small air gap separates the dielectric and the waveguide. From experience, these small air gaps break down quite easily.

Conclusion

By using slabs of dielectric in a square waveguide, quarter-wave plates may be built to operate over a wide frequency band. It is shown theoretically that they can be made to operate over a 2 to 1 frequency band and there is no reason that this could not be expanded considerably. In practice, one has to consider the higher order modes that this plate would generate over such bandwidths. Experimentally, quarter-wave plates were made to operate over a standard waveguide bandwidth.

Complete design information is given for quarter-wave plates made of polystyrene. The matching problem is solved by tapering the ends for which calculations are given. The experimental results closely follow the calculated values. Experimental results also show that the quarter-wave plate action is very closely the same in circular as in square waveguide.

Acknowledgment

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